Announcements

1) HW due today

2) New HW up, due
next Thursday

We were proving:

Theorem: (connected sets in $\mathbb{R}$)

$S \subseteq \mathbb{R}$ (with the standard metric)

is connected if and only if

whenever $x, y \in S$, $x < y$,

then $\forall$ z with $x < z < y$,

$$\boxed{z \in S}$$

finish proof

we proved $\Rightarrow$

Started to prove $\Leftarrow$

What we proved last time allows us to assume that if $S = T \cup V$,

then $\exists\ x, y \in T$ and $z \in V$,

$$x < z < y.$$

Let $\alpha = \inf\{w \in T : w > z\}$

Note that by our property, since $z, y \in S$ and $z \leq \alpha \leq y$, $\alpha \in S$.

2 cases

$\underline{\alpha = z}$ If so, then $z \in \overline{T}$. But $z \in V$, so $\overline{T} \cap V \neq \emptyset$.

$\underline{\alpha \neq z}$ This implies $z < \alpha$.

By our property, if $z < w < \alpha$, then $w \in S$. This implies $[z, \alpha) \subseteq S$. But since $\alpha = \inf\{w \in T \mid w > z\}$, $[z, \alpha) \subseteq V$.

Since $[z,\alpha) = [z,\alpha] \subseteq \overline{V}$,

$\alpha \in \overline{V}$. By definition,

$\alpha \in \overline{T}$, so either

$V \cap \overline{T}$ or $T \cap \overline{V}$ is

nonempty depending on

whether $\alpha \in V$ or $\alpha \in T$

(remember $V \cup T = S$ and $\alpha \in S$).

This shows S is connected. $\square$

Corollary: $S \subseteq \mathbb{R}$ (usual metric) is connected if and only if S is an interval.

proof: $\Leftarrow$ If S is an interval, then $\forall\, x, y \in S$, if $x < z < y$, $z \in S$. By the previous theorem, S is connected.

$\Rightarrow$ If S is connected, then by the theorem, $\forall\, x, y \in S$, if $x < z < y$, $z \in S$

- If $\sup(S)$ and $\inf(S)$ do not exist, then S is unbounded. By our characterization, $S = \mathbb{R}$.

- If $\sup(S) = \alpha$ does exist, then if $\inf(S)$ does not exist, then $S = (-\infty, \alpha)$ or $(-\infty, \alpha]$ by the characterization

- Similarly, if $\inf(S) = \beta$ exists, but $\sup(S)$ does not exist, $S = (\alpha, \infty)$ or $[\alpha, \infty)$

— finally, if α and β both exist, S is one of

(α, β), $[\alpha, \beta)$, $(\alpha, \beta]$,

or $[\alpha, \beta]$ $\square$

Back to the Cantor Set

$$C_n = [0,1] \setminus \left(\bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right) \right)$$

$$\bigcap_{n=1}^{k} C_n = k^{th} \text{ stage Cantor construction}$$

Cantor set $C = \bigcap_{n=1}^{\infty} C_n$

Check properties for Cantor

Set

Is C closed?

C_n is the union of finitely many closed intervals.

This means that each C_n is closed since finite unions of closed sets are closed.

Then $C = \bigcap_{n=1}^{\infty} C_n$ is closed since infinite intersections of closed sets are closed.

Is C compact?

C is bounded ($C \subseteq [0,1]$) and we just showed C is closed, so C is compact by Heine-Borel.

Is C connected?

$0 \in C$ and $1 \in C$. Since $C \neq [0,1]$, C is not connected.

Is C open?

The interval $\left(\frac{1}{3^n}, \frac{2}{3^n}\right) \in C^c$ for all $n \geq 1$. This gives a sequence of points

$$x_n \in \left(1/3^n, \frac{2}{3^n}\right), \quad \lim_{n \to \infty} x_n = 0$$

(your choice of x_n doesn't matter as long as $x_n \in \left(\frac{1}{3^n}, \frac{2}{3^n}\right)$)

This shows $0 \in \overline{C^c}$.

But $0 \in C$, so C^c is not closed. Therefore, C is not open.

The Hausdorff Dimension

Let $S \subseteq X$, X a metric space.

Let $\{B(x, r_i)\}_{i \in I}$ be any open cover of S by open balls of radius r_i

Note r_i may be different for each $i \in I$

Let $\mathcal{B}$ be the collection of all such covers of S and let

$$C_d(S) = \inf_{B \in \mathcal{B}} \left\{ \sum_{i \in I} r_i^d \right\}$$

The Hausdorff dimension $\dim_H(S)$ of S is given by

$$\boxed{\dim_H(S) = \inf\{d \geq 0 \mid C_d(S) = 0\}}$$

Some Hausdorff Dimensions

$$\dim_H(\mathbb{R}^n) = n$$

(generalizes vector space dimension)

$$\dim_H(\mathbb{Q}) = \dim_H(\{\text{point}\}) = 0$$

$$\dim_H(C) = \frac{\ln(2)}{\ln(3)}$$

If you're interested in this:

Survey article on CTools under "Resources"